Period Tripling and Chaos in the Dynamic Behavior of Directly Modulated Diode Lasers

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Abstract—This paper presents the relevance of period-tripling behavior that has recently been found in different experimental studies of directly modulated laser diodes. Applying different numerical techniques to the rate equation model of the laser diode, among which we highlight the continuation method to calculate the unstable solutions of the system, we show that period-tripling behavior appears and disappears in two tangent bifurcations. Therefore, the period-three solutions form a closed bifurcation curve called isola. In between these two tangent bifurcations, the period-three solution coexists with the chaotic attractor reached by a period-doubling cascade, giving rise to a hysteresis loop in the deterministic case. Also, we have found that a boundary crisis might be behind the chaotic behavior that is observed for the highest values of the modulation index. The effects of random noise fluctuations in the laser diode dynamics are also studied. Langevin noise sources are included in the rate equation and appropriate stochastic integration methods have been used. The route to chaos that we have obtained points out the relevant role that noise has in achieving agreement between numerical studies and experimental results that have been published. The introduction of noise has been proved to be of major importance in determining the system behavior in the regions of the coexistence of solutions.

Index Terms—Bifurcation, chaos, dynamics, numerical analysis, semiconductor lasers, stochastic differential equations.

I. INTRODUCTION

The dynamics of lasers have been extensively studied since it was shown that the rate equation system describing its behavior is isomorphic to the Lorenz equations. This fact extended the use of lasers in order to observe experimentally the nonlinear behavior that was predicted by the Lorenz model. Among these, lasers with modulated parameters are the most widely used. Due to their practical interest in a variety of demanding applications, a great number of studies have been devoted to directly modulated laser diodes.

The first theoretical results using the rate equation model that were performed indicated that, as the control parameter is varied, laser diodes present a period-doubling route to chaos [1], [2]. Soon after, the experimental data available reported only the observation of a few period-doubling bifurcations.

After these, period halving takes place, and the response of the laser returns to follow the modulating frequency. This suggested that unaccounted effects in the rate equation model as well as random fluctuations could be responsible for the truncation of the period-doubling sequence. An early study of the effect of noise on laser dynamics reported that a noisy laser model is unable to produce higher order bifurcations [3]. Other studies demonstrated that random fluctuations provide the ability to predict the occurrence of a bifurcation before it actually appears due to the existence of the so-called noise precursors in the frequency spectrum developed by Wiesenfeld [4]. The precursors corresponding to period-doubling bifurcations have been demonstrated in laser diodes both theoretically and experimentally [5].

On the other hand, other experiments have shown that the response of the diode laser is more complex, showing the appearance of a period-tripling behavior as the laser is modulated at high levels of injection current. To the best of our knowledge, only two recent studies address the nonlinear dynamics in directly modulated lasers experimentally showing period-tripling behavior [6], [7]. Only one of them, using a 1.5-μm distributed feedback (DFB) laser, reports chaotic behavior, finding period-doubling, period-quadrupling, and period-tripling stages before the onset of chaos [6]. The other study compares the behavior of two different multiquantum-well (MQW) lasers, a Fabry–Perot (FP) and a DFB, finding only period-doubling and period-tripling behaviors [7]. Sarkovki’s theorem states that any system that exhibits a period-tripling behavior must also present solutions with infinite period, in other words, the system has chaotic solutions [8].

In this paper, we show a possible explanation for the route to chaos that was observed in DFB lasers, which may also explain its experimental absence in MQW-DFB lasers [6], [7]. The two key elements are the coexistence of the period-tripling solution with the period-doubling cascade and the random noise. The joint effect that they have in the dynamics of directly modulated laser diodes can help to understand the observed behavior. The use of a continuation method has been a determinant in finding the crisis phenomena in laser diodes since it allowed us to follow the solutions of the system beyond the bifurcation points and provide the unstable solutions of the laser [9]. This crisis phenomenon is similar to those observed in earlier experiments with CO₂ lasers [10]. The exposition of this paper is the following. We start by presenting the laser diode model in Section II, along with a brief presentation of the mathematical tools employed.

Manuscript received February 24, 1998; revised July 1, 1998. This work was supported in part by the European Commission under ERBFMRXCT980223 Training and Mobility Researchers FALCON Project and by the CICYT Spanish Commission under Project TIC96–1415–E.

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Publisher Item Identifier S 0018–9197(98)07189–9.
in our study. The dynamic behavior of the laser diode in the absence of noise is studied in Section III, showing the regions of coexistence of solutions giving rise to hysteresis loops and crisis phenomena. The introduction of the random noise fluctuations in the rate equation system is then studied by turning the Langevin noise sources on. The results of the stochastic rate equations are presented in Section IV, followed by the conclusions in Section V.

II. THEORETICAL MODEL

The laser diode has been described using the rate equation model, composed by two coupled nonlinear ordinary differential equations that account for the time evolution of the carrier and photon densities in the lasing region. Assuming a constant carrier lifetime, the dimensionless form of these equations results in the following set:

\[
\frac{dn}{dt} = i - n - \left( 1 + ks \right) s + F_n(\tau) \quad (1.1)
\]

\[
i = \dot{i}_n + \dot{i}_m \sin(\omega_m \tau) \quad (1.2)
\]

\[
\frac{ds}{dt} = T \left[ S_1 \left( 1 + ks \right) s - s + S_2 m \right] + F_s(\tau). \quad (1.3)
\]

As a result of the normalization process, the coefficients appearing in the normalized system are the quotient between the carrier lifetime and the photon lifetime \( T = \tau_n / \tau_p \), the normalized gain compression factor \( k = \epsilon / (g_0 \cdot \tau_n) \), the normalized transparency carrier density \( n_t = N_0 / N_{th} \), and the normalized spontaneous emission coupling factor \( S_2 = \beta \cdot S_1 = \beta \cdot \gamma \tau_p g_0 N_{th} \). The factor appearing on the right-hand side of (1.1) is the normalized injection current used to introduce the forcing term composed of a constant bias current \( \dot{i}_n \) and a sinusoidal time-dependent modulation with normalized angular frequency \( \omega_m \) and period \( T_m \). The depth of the modulation is expressed through the modulation index, \( m \), which is defined as \( m = \dot{i}_m / (\dot{n}_n - \dot{n}_{th}) \). Since we are interested in obtaining nonlinear behavior from the laser, the modulation frequency has been taken as twice the relaxation oscillation frequency. The rate equation model has been normalized to have dimensionless unit order variables. The terms \( F_n(\tau) \) and \( F_s(\tau) \) are Langevin noise sources introducing the random fluctuations of the carriers and photons, respectively.

The rate equation system can be written in a more convenient way using a matrix notation, leading to the following expression of the system:

\[
\dot{x} = R[x(\tau), \tau, m] + L[x(\tau), \tau]
\]

where \( x \) is a column vector that contains the carrier and photon densities, \( R[\cdot] \) is the deterministic part on the right-hand side, and \( L[\cdot] \) is a column vector with the Langevin noise sources. The explicit time dependence of the deterministic term, due to the modulation term in the carrier rate equation, points out that we are dealing with a nonautonomous dynamic system. We have also included the modulation index dependence since we are going to study the laser diode behavior when the modulation index is varied. The time evolution of the deterministic rate equation system has been shown to present periodic solutions, with its period relating to that of the modulation. In mathematical terms, a periodic solution is said to define a “limit cycle” or “periodic orbit” and satisfies the relation

\[
x(\tau) = x(\tau + T_{per}).
\]

This relation states that any periodic solution coincides with itself after a fixed time interval \( T_{per} \) has elapsed. The time needed to repeat itself is the period of the solution. By studying the evolution of the dynamic behavior as one of the parameters of the system was changed, we obtained the system’s bifurcation diagram. These diagrams are constructed by choosing an appropriate Poincaré section and recording the values of the system variables whenever the solution intersects the section. In our case, since we have taken a fixed phase of the sinusoidal periodic forcing to be our Poincaré section, recording the intersections is equivalent to sampling the carrier and photon densities at integral multiples of the modulation period. The bifurcation diagram is the result of plotting the recorded values against the parameter of the system that is varied, which in our case is the modulation index.

In order to follow the unstable solutions of the system, we have also used a continuation method. Continuation methods take advantage of previous knowledge of a solution at a given value of the modulation index, given by \( m_{op} \), in order to find the solution at a slightly perturbed value of the modulation index, \( m_1 = m_{op} + \delta m \). The direction in which the continuation parameter is perturbed \( (\delta m) \) is guided by the Jacobian of the system with respect to the modulation index, providing the algorithm with the tangent vector at the known solution. A predictor step makes use of this vector to provide an estimation of the new solution by natural parameter continuation. At singularities such as folds or bifurcation points, arclength continuation is used to allow the method to find the estimation of the solution [9]. A corrector step, which refines the estimation, follows the predictor step. The desired accuracy of the estimation is achieved using a Newton method. By successive application of the method, it allows us to follow the evolution of any desired solution of the system as the modulation index is varied.

III. DETERMINISTIC RESULTS

The first approach used to study the system’s dynamics has been to find the solutions to the deterministic system. The system of equations that must be solved is that given by (1) when the Langevin noise sources are eliminated. The laser diode parameters have been given standard values: \( \tau_p = 6E \cdot 12s, \tau_n = 3E \cdot 9s, V = 2.5E \cdot 10cm^{-3}, \gamma = 0.4, g_0 = 5E \cdot 7cm^3s^{-1}, \epsilon = 0.6E \cdot 17cm^3, N_0 = 1.5E18 cm^{-3}, \) and \( \beta = 5E - 5 \) which correspond to the photon lifetime, the carrier lifetime, the volume of the active layer, the optical confinement factor, the linear gain coefficient, the gain compression factor, the carrier density needed to achieve transparency, and the spontaneous emission factor, respectively. Our normalized constants result in the following: \( T = 500, k = 0.004, n_t = 0.6429, S_1 = 2.8, \) and \( S_2 = 1.4 \cdot 10^{-4}. \)
has been taken as a bifurcation parameter, and periodic saddle cycle is causing the coexistence of the solutions in tangent bifurcations has also been evident from the figure. The period three solution appeared for the first time is also shown to occur in CO₂ lasers [12], both the stable and unstable branches of the period-three solution meet each other annihilating themselves. The result is that the period-three solutions create an isola (closed bifurcation curve). While isolas were shown to exist in CO₂ lasers when the damping rate was increased beyond a realistic value [11], this is the first time that isolas have been reported in laser diodes, which, due to some characteristic effects such as gain compression and a larger spontaneous emission, have a larger damping rate.

2) A second boundary crisis is responsible for the appearance of a new chaotic attractor. The chaotic behavior that is observed experimentally straight after the period-three stage can be explained by a second boundary crisis, which gives rise to the chaotic attractor. Therefore, although the studies in which period-tripling appeared primarily for CO₂ lasers [12], it cannot exclude the appearance of chaotic behavior as this behavior can appear through crisis phenomena.

In Fig. 1, we present the bifurcation diagram where the modulation index \( m \) has been taken as a bifurcation parameter and the Poincaré section is located at a fixed phase of the sinusoidal modulation. In this figure, we have included the stable cycles (with dots), calculated using standard numerical algorithms, and the unstable saddle cycles (straight lines), which were found using a continuation method. The first thing that one notices is the appearance of a period-doubling cascade as the modulation index is increased, eventually reaching chaotic behavior. This is the well-known phenomenon of the period-doubling route to chaos in modulated lasers as was found in earlier studies of their dynamic behavior [1], [2]. Nevertheless, one can also appreciate that the chaotic behavior disappears suddenly when the modulation index is increased beyond the value \( m = 5.6 \). From that value on, the period three behavior is the only solution of the system. The question about at which modulation index level the period three solution appeared for the first time is also evident from the figure. The period three solution appears in a tangent bifurcation point at \( m = 4.5 \) causing the coexistence of the chaotic behavior with the period three solution. The appearance of solutions in tangent bifurcations has also been shown to occur in CO₂ lasers [11] resulting in coexistence and crisis phenomena. In our case, the coexistence of the period-three cycle and the period doubling sequence gives rise to a hysteresis loop, marked with arrows in Fig. 1. This loop starts at the modulation index level at which the tangent bifurcation appears and ends when the chaotic attractor is annihilated in a collision with the period-three saddle cycle in a boundary crisis. A crisis occurs when a chaotic attractor touches a coexisting unstable fixed point or periodic orbit, leading to sudden changes in the chaotic attractor. As we have just shown, in a laser diode the unstable periodic orbit is a period-three saddle cycle and chaotic attractor disappears after colliding with it.

However, the unstable period-three orbits are not the only unstable solutions of the system. Every time a period-doubling bifurcation takes place, a stable \( T \)-periodic solution gives rise to a \( 2T \)-periodic stable cycle and a \( T \)-periodic saddle (a straight line in Fig. 1). As the modulation index is increased, the chaotic attractor progressively increases its size by successively embedding the saddle cycles created previously by the Feigenbaum sequence. The incorporation of saddle cycles provides the strange attractor with its characteristic stretching and folding behavior. The embedding process is presented in Fig. 2 by plotting the state space diagrams at four different values of the modulation index. Fig. 2(a) plots the state space diagram when the modulation index level is set at 4.6. The period-doubling cascade produces a stable period-eight cycle, represented by eight spots. In that figure, the spots left on the Poincaré plane by the coexisting \( T_m \), \( 2 \cdot T_m \), and \( 3 \cdot T_m \) periodic saddle cycles also appear. As the modulation index is increased, the embedding process is clearly observed. The state space diagram at \( m = 4.7 \) is plotted in Fig. 2(b). At this level, it can be appreciated that the period-four saddle cycle has already been embedded into the strange attractor by observing that the four spots left by this cycle fall within the continuous traces (chaotic attractor). The chaotic character of the solution is confirmed by the stability analysis, which provides a positive Lyapunov exponent [8]. The diagrams at successive modulation indexes, at 4.8 and 4.9, are given in Fig. 2(c) and (d), respectively. For these two levels of the modulation index, the \( 2 \cdot T_m \)-periodic saddle cycle is embedded at 4.8 and the \( T_m \) periodic saddle cycle is finally embedded at 4.9. This phenomenon, in which an unstable solution is embedded by a chaotic attractor is known as interior crisis [10].

The dynamic behavior of the laser diode for a larger modulation index level has also been studied. In Fig. 3 we present the bifurcation diagram for the modulation index level ranging between 7 and 12. This figure reveals two important facts about the dynamic behavior of laser diodes.

1) The period-three cycle disappears in an inverse tangent bifurcation. Instead of starting a new period-doubling cascade (going through \( 6 \cdot T_m \), \( 12 \cdot T_m \), \( 24 \cdot T_m \), ..., as in CO₂ lasers [12]), both the stable and unstable branches of the period-three solution meet each other annihilating themselves. The result is that the period-three solutions create an isola (closed bifurcation curve). While isolas were shown to exist in CO₂ lasers when the damping rate was increased beyond a realistic value [11], this is the first time that isolas have been reported in laser diodes, which, due to some characteristic effects such as gain compression and a larger spontaneous emission, have a larger damping rate.

2) A second boundary crisis is responsible for the appearance of a new chaotic attractor. The chaotic behavior that is observed experimentally straight after the period-three stage can be explained by a second boundary crisis, which gives rise to the chaotic attractor. Therefore, although the studies in which period-tripling appeared primarily for CO₂ lasers [12], it cannot exclude the appearance of chaotic behavior as this behavior can appear through crisis phenomena.
Fig. 2. State space diagrams of the carrier density versus photon density at (a) $m = 4.6$, (b) $m = 4.7$, (c) $m = 4.8$, and (d) $m = 4.9$. The saddle cycles are marked with (*) $\cdot T_m$, (×) $\cdot 2 \cdot T_m$, and (+) $\cdot T_m$-periodic solutions.

The theory of noise precursors developed by Wiesenfeld [4] introduced a characteristic phenomenon of noisy systems suffering period-doubling bifurcations, called the Virtual Hopf phenomenon, verified later experimentally and numerically in a laser diode [5].

IV. STOCHASTIC RESULTS

We include noise in our normalized rate equation system (1) based on Langevin dynamics. We obtain normalized multiplicative noise sources and the integration of the rate equations becomes a rather delicate matter. Since we want to obtain the bifurcation diagram including noise, we are interested in computing the time evolution of the trajectories for a given initial condition. We must, therefore, use a strong approximation numerical method such as a Heun scheme. The bifurcation diagram is presented in Fig. 4, where one can observe that noise truncates the route to chaos via period-doubling bifurcations. The role of noise is to allow the system to explore the phase space surroundings of the current solution. Therefore, whenever the deterministic system exhibits a coexistence of attractors that are located close to each other in the phase space, we must include the random

Fig. 3. Bifurcation diagram of the carrier density versus modulation index ranging from $m = 7$ to $m = 12$. This figure is labeled as in Fig. 1.

On the other hand, the studies performed on laser diodes blamed random noise for the absence of chaotic behavior in experimental setups [3]. Several studies have been devoted to clear up the role of noise in period-doubling bifurcations.
solutions in our analysis of the dynamics. Moreover, spectral analysis of the behavior at several modulation indices reveals that the route to chaos that we obtain is via period doubling, period quadrupling, and period tripling [9], in agreement with the experimental observation [6]. Since at high levels of modulation index there is no other solution close to the chaotic attractor, random noise does not eliminate this behavior (Fig. 4).

From our study, it is now possible to suggest the mechanism behind the absence of chaotic behavior in the latest experiment that has been performed on DFB-MQW lasers [7]. As we have seen, the chaotic attractor can be hidden by the coexistence with the period-tripling behavior and the effect of noise. The fact that the damping term influences the period-three behavior [11] leads us to believe that differences in the damping terms among lasers are responsible for the presence or absence of chaotic behavior [6], [7]. In any case, random noise fluctuations are of fundamental importance to achieve agreement between experimental and numerical results.

V. CONCLUSIONS

This paper presents the relevance of the period-three solution that appears in the dynamics of directly modulated laser diodes. We show that this solution, reported experimentally, appears and disappears due to tangent bifurcations at different modulation index values. We show that this solution coexists with the period-doubling route to chaos, giving rise to a hysteresis loop and crisis phenomena in the deterministic case. The period-three solution has also been shown to form an isola in the bifurcation diagram, appearing and disappearing by tangent bifurcations. The joint effect of this period-tripling behavior with random noise is to effectively truncate the period-doubling sequence that was reported in earlier studies of directly modulated laser diode dynamics.

The chaotic behavior that has been observed in laser diodes can then be explained by the appearance of a new chaotic attractor due to a boundary crisis in which the unstable branches of the period-tripling solution are involved. In fact, despite the increased level of damping and noise in laser diodes, chaotic dynamics are not eliminated. Our results point out that the introduction of noise into the rate equation system is necessary in order to obtain an agreement of the numerical route to chaos with the observed behavior.

ACKNOWLEDGMENT

The authors would like to thank S. Lewis for her careful review of the manuscript.

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